Projective Geometry
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Projective Geometry

Renaissance painters created it to represent three-dimensional reality in two dimensions. Their invention finally transcended Euclidean geometry and today forms an integral part of physics

by Morris Kline

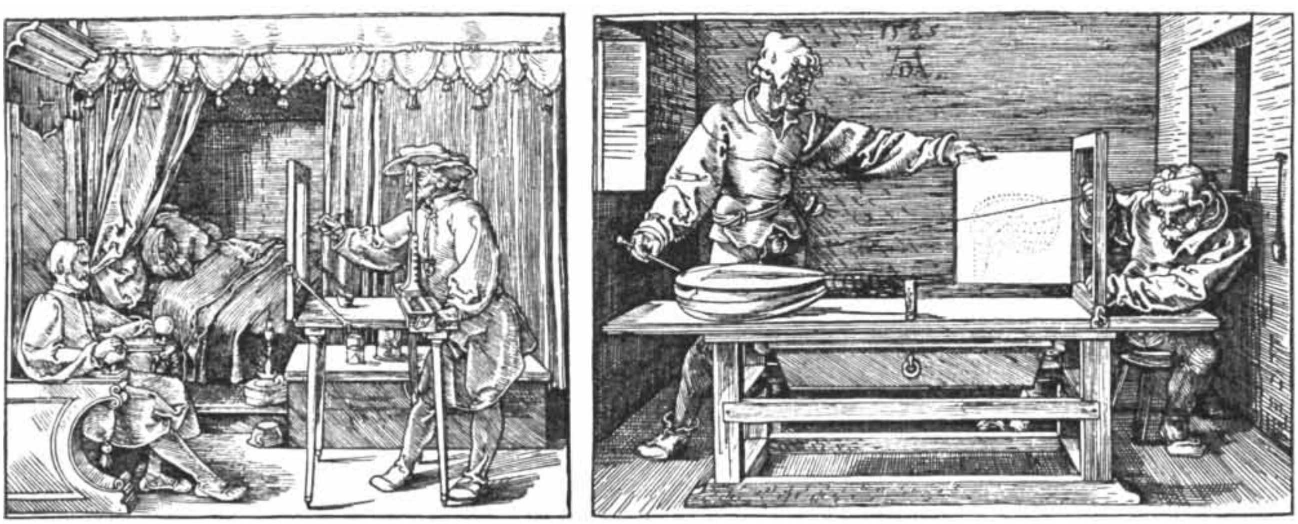
In the house of mathematics there are many mansions and of these the most elegant is projective geometry. The beauty of its concepts, the logical perfection of its structure and its fundamental role in geometry recommend the subject to every student of mathematics.

Projective geometry had its origins in the work of the Renaissance artists. Medieval painters had been content to express themselves in symbolic terms. They portrayed people and objects in a highly stylized manner, usually on a gold background, as if to emphasize that the subject of the painting, generally religious, had no connection with the real world. An excellent example, regarded by critics as the flower of medieval painting, is Simone Martini's "The Annunciation." With the Renaissance came not only a desire to paint realistically but also a revival of the Greek doctrine that the essence of nature is mathematical law. Renaissance painters struggled for over a hundred years to find a mathematical scheme which would enable them to depict the three-dimensional real world on a two-dimensional canvas. Since many of the Renaissance painters were architects and engineers as well as artists, they eventually succeeded in their objective. To see how well they succeeded one need only compare Leonardo da Vinci's "Last Supper" with Martini's "Annunciation" [*see opposite page*].

The key to three-dimensional representation was found in what is known as the principle of projection and section. The Renaissance painter imagined that a ray of light proceeded from each point in the scene he was painting to one eye. This collection of converging lines he called a projection. He then imagined that his canvas was a glass screen interposed between the scene and the eye. The collection of points where the lines of the projection intersected the glass screen was a "section." To achieve realism the painter had to reproduce on canvas the section that appeared on the glass screen.

Two woodcuts by the German painter Albrecht Dürer illustrate this principle of projection and section [*see below*]. In "The Designer of the Sitting Man" the artist is about to mark on a glass screen a point where one of the light rays from the scene to the artist's eye intersects the screen. The second woodcut, "The Designer of the Lute," shows the section marked out on the glass screen.

Of course the section depends not only



WOODCUTS by Albrecht Dürer illustrate the principle of projection and section. In the first woodcut the artist is about to mark the point at which a light ray from the scene to his eye intersects a glass screen. In the second a scene is marked out on the screen.

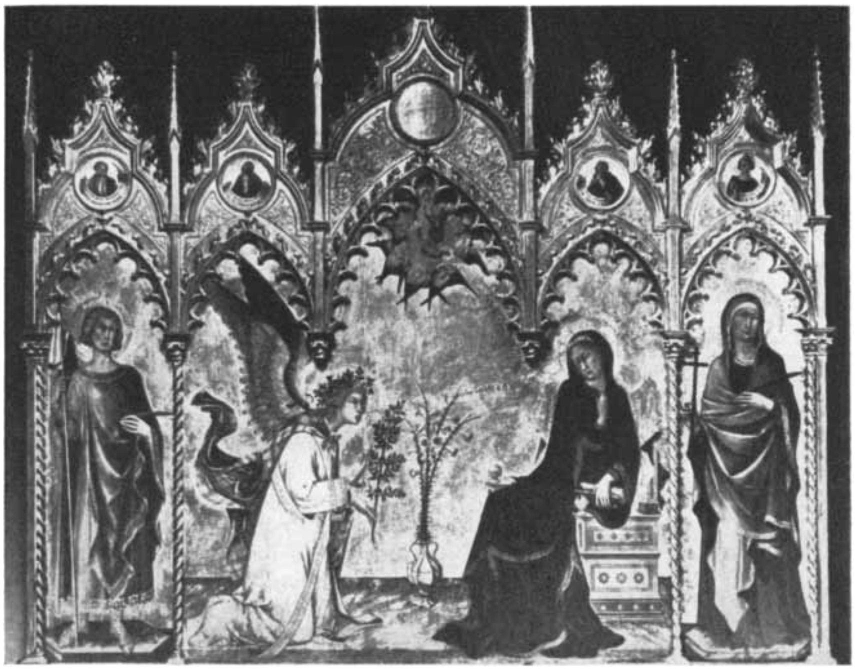
upon where the artist stands but also where the glass screen is placed between the eye and the scene. But this just means that there can be many different portrayals of the same scene. What matters is that, when he has chosen his scene, his position and the position of the glass screen, the painter's task is to put on canvas precisely what the section contains. Since the artist's canvas is not transparent and since the scenes he paints sometimes exist only in his imagination, the Renaissance artists had to derive theorems which would specify exactly how a scene would appear on the imaginary glass screen (the location, sizes and shapes of objects) so that it could be put on canvas.

The theorems they deduced raised questions which proved to be momentous for mathematics. Professional mathematicians took over the investigation of these questions and developed a geometry of great generality and power. Let us trace its development.

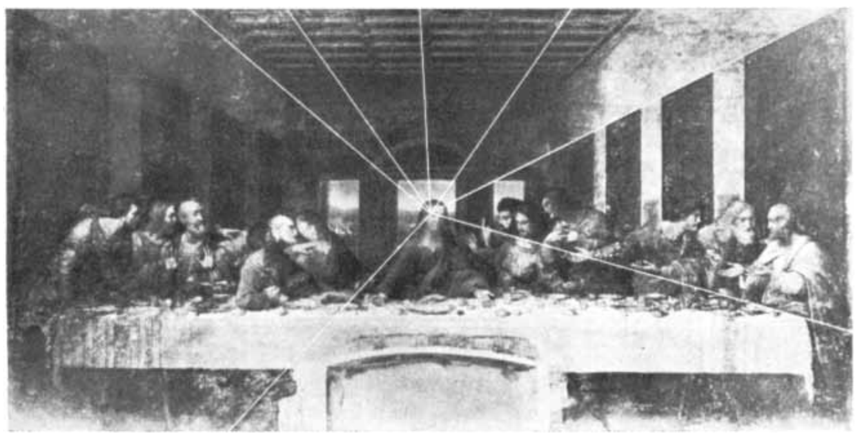
Suppose that a square is viewed from a point somewhat to the side [*Figure 1*]. On a glass screen interposed between the eye and the square, a section of its projection is not a square but some other quadrilateral. Thus square floor tiles, for instance, are not drawn square in a painting. A change in the position of the screen changes the shape of the section, but so long as the position of the viewer is kept fixed, the impression created by the section on the eye is the same. Likewise various sections of the projection of a circle viewed from a fixed position differ considerably—they may be more or less flattened ellipses—but the impression created by all these sections on the eye will still be that created by the original circle at that fixed position.

To the intellectually curious mathematicians this phenomenon raised a question: Should not the various sections presenting the same impression to the eye have some geometrical properties in common? For that matter, should not sections of an object viewed from different positions also have some properties in common, since they all derive from the same object? In other words, the mathematicians were stimulated to seek geometrical properties common to all sections of the same projection and to sections of two different projections of a given scene. This problem is essentially the one that has been the chief concern of projective geometers in their development of the subject.

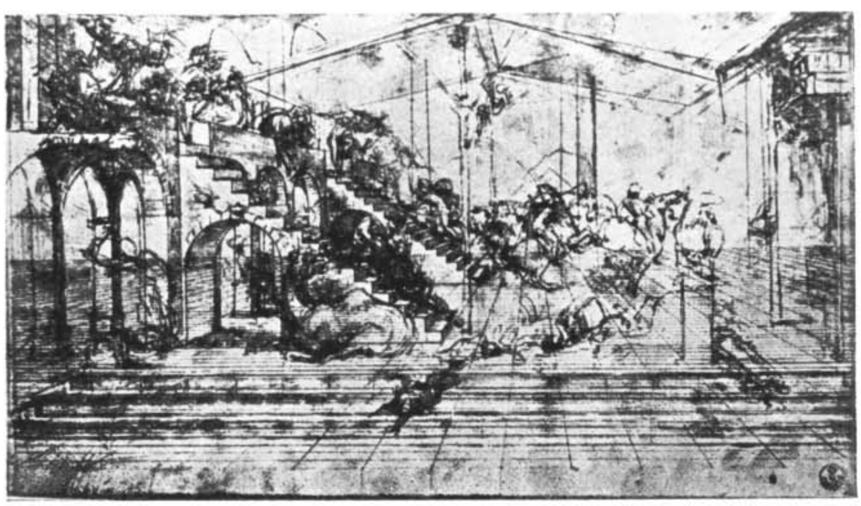
It is evident that, just as the shape of a square or a circle varies in different



THE ANNUNCIATION by Simone Martini is an outstanding example of the flat, stylized painting of the medieval artists. The figures were symbolic and framed in a gold background.



THE LAST SUPPER by Leonardo da Vinci utilized projective geometry to create the illusion of three dimensions. Lines have been drawn on this reproduction to a point at infinity.



DRAWING by da Vinci, made as a study for his painting "The Adoration of the Magi," shows how he painstakingly projected the geometry of the entire scene before he actually painted it.

sections of the same projection or in different projections of the figure, so also will the length of a line segment, the size of an angle or the size of an area. More than that, lines which are parallel in a physical scene are not parallel in a painting of it but meet in one point; see, for example, the lines of the ceiling beams in da Vinci's "Last Supper." In other words, the study of properties common to the various sections of projections of an object does not seem to lie within the province of ordinary Euclidean geometry.

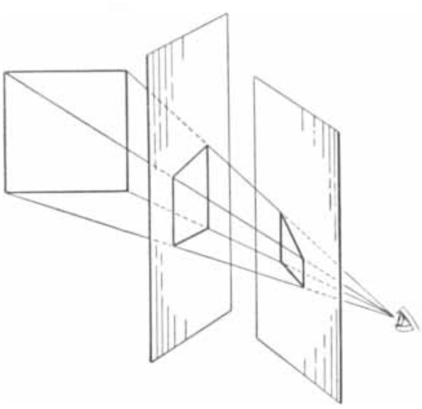


Figure 1 (see text)

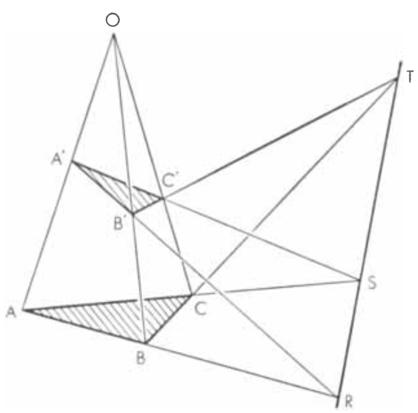


Figure 2

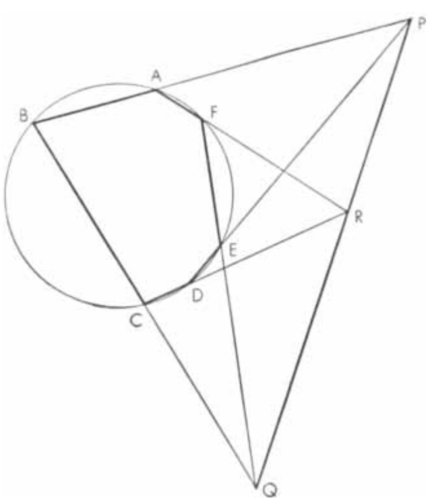


Figure 3

Yet some rather simple properties that do carry over from section to section can at once be discerned. For example, a straight line will remain a line (that is, it will not become a curve) in all sections of all projections of it; a triangle will remain a triangle; a quadrilateral will remain a quadrilateral. This is not only intuitively evident but easily proved by Euclidean geometry. However, the discovery of these few fixed properties hardly elates the finder or adds appreciably to the structure and power of mathematics. Much deeper insight was required to obtain significant properties common to different sections.

The first man to supply such insight was Gérard Desargues, the self-educated architect and engineer who worked during the first half of the 17th century. Desargues's motivation was to help the artists; his interest in art even extended to writing a book on how to teach children to sing well. He sought to combine the many theorems on perspective in a compact form, and he invented a special terminology which he thought would be more comprehensible than the usual language of mathematics.

His chief result, still known as Desargues's theorem and still fundamental in the subject of projective geometry, states a significant property common to two sections of the same projection of a triangle. Desargues considered the situation represented here by two different sections of the projection of a triangle from the point O [*Figure 2*]. The relationship of the two triangles is described by saying that they are perspective from the point O. Desargues then asserted that each pair of corresponding sides of these two triangles will meet in a point, and, most important, these three points will lie on one straight line. With reference to the figure, the assertion is that AB and $A'B'$ meet in the point R; AC and $A'C'$ meet in S; BC and $B'C'$ meet in T; and that R, S and T lie on one straight line. While in the case stated here the two triangle sections are in different planes, Desargues's assertion holds even if triangles ABC and $A'B'C'$ are in the same plane, *e.g.*, the plane of this paper, though the proof of the theorem is different in the latter case.

The reader may be troubled about the assertion in Desargues's theorem that each pair of corresponding sides of the two triangles must meet in a point. He may ask: What about a case in which the sides happen to be parallel? Desargues disposed of such cases by invoking the mathematical convention that any set of parallel lines is to be regarded as having a point in common, which the student is often advised to think of as being at infinity—a bit of advice which essentially amounts to answering a question by not answering it. However, whether or not one can visualize this point at infinity is immaterial. It is logically possible to agree that parallel lines are to be regarded as having a point in common, which point is to be distinct from the usual, finitely located points of the lines considered in Euclidean geometry. In addition, it is agreed in projective geometry that all the intersection points of the different sets of parallel lines in a given plane lie on one line, sometimes called the line at infinity. Hence even if each of the three pairs of corresponding sides of the triangles involved in Desargues's theorem should consist of parallel lines, it would follow from our agreements that the three points of intersection lie on one line, the line at infinity.

These conventions or agreements not only are logically justifiable but also are recommended by the argument that projective geometry is concerned with problems which arise from the phenomenon of vision, and we never actually see parallel lines, as the familiar example of the apparently converging railroad tracks remind us. Indeed, the property of parallelism plays no role in projective geometry.

At the age of 16 the precocious French mathematician and philosopher Blaise Pascal, a contemporary of Desargues, formulated another major theorem in projective geometry. Pascal asserted that if the opposite sides of any hexagon inscribed in a circle are prolonged, the three points at which the extended pairs of lines meet will lie on a straight line [*Figure 3*].

As stated, Pascal's theorem seems to have no bearing on the subject of projection and section. However, let us visualize a projection of the figure involved in Pascal's theorem and then visualize a section of this projection [*Figure 4*]. The projection of the circle is a cone, and in general a section of this cone will not be a circle but an ellipse, a hyperbola, or a parabola—that is, one of the curves usually called a conic section. In any conic section the hexagon in the original circle will give rise to a corresponding hexagon. Now Pascal's theorem asserts that the pairs of opposite sides of the new hexagon will meet on one straight

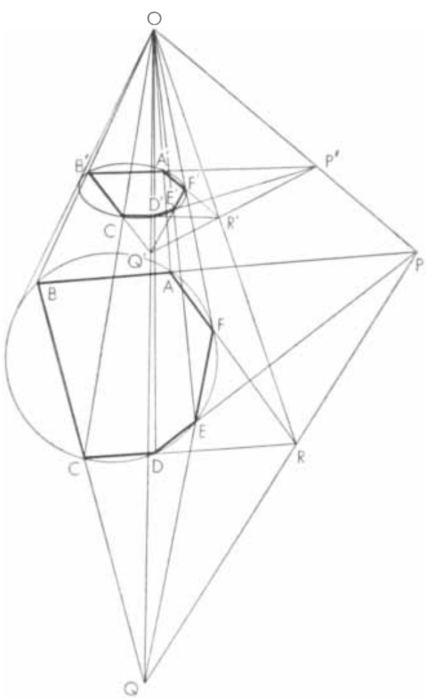


Figure 4

line which corresponds to the line derived from the original figure. Thus the theorem states a property of a circle which continues to hold in any section of any projection of that circle. It is indeed a theorem of projective geometry.

It would be pleasant to relate that the theorems of Desargues and Pascal were immediately appreciated by their fellow mathematicians and that the potentialities in their methods and ideas were eagerly seized upon and further developed. Actually this pleasure is denied us. Perhaps Desargues's novel terminology baffled mathematicians of his day, just as many people today are baffled and repelled by the language of mathematics. At any rate, all of Desargues's colleagues except René Descartes exhibited the usual reaction to radical ideas: they called Desargues crazy and dismissed projective geometry. Desargues himself became discouraged and returned to the practice of architecture and engineering. Every printed copy of Desargues's book, originally published in 1639, was lost. Pascal's work on conics and his other work on projective geometry, published in 1640, also were forgotten. Fortunately a pupil of Desargues, Philippe de la Hire, made a manuscript copy of Desargues's book. In the 19th century this copy was picked up by accident in a bookshop by the geometer Michel Chasles, and thereby the world learned the full extent of Desargues's major work. In the meantime most of Desargues's and Pascal's discoveries had had to be remade independently by 19th-century geometers.

Projective geometry was revived through a series of accidents and events almost as striking as those that had originally given rise to the subject. Gaspard Monge, the inventor of descriptive geometry, which uses projection and section, gathered about him at the Ecole Polytechnique a host of bright pupils, among them Sadi Carnot and Jean Poncelet. These men were greatly impressed by Monge's geometry. Pure geometry had been eclipsed for almost 200 years by the algebraic or analytic geometry of Descartes. They set out to show that purely geometric methods could accomplish more than Descartes's.

It was Poncelet who revived projective geometry. As an officer in Napoleon's army during the invasion of Russia, he was captured and spent the year 1813-14 in a Russian prison. There Poncelet reconstructed, without the aid of any books, all that he had learned from Monge and Carnot, and he then proceeded to create new results in projective geometry. He was perhaps the first mathematician to appreciate fully that this subject was indeed a totally new branch of mathematics. After he had reopened the subject, a whole group of French and, later, German mathematicians went on to develop it intensively.

One of the foundations on which they built was a concept whose importance had not previously been appreciated. Consider a section of the projection of a line divided by four points [*Figure 5*]. Obviously the segments of the line in the section are not equal in length to those of the original line. One might venture that perhaps the ratio of two segments, say $A'C'/B'C'$, would equal the corresponding ratio AC/BC. This conjecture is incorrect. But the surprising fact is that the ratio of the ratios, namely $(A'C'/C'B')/(A'D'/D'B')$, will equal $(AC/CB)/(AD/DB)$. Thus this ratio of ratios, or cross ratio as it is called, is a projective invariant. It is necessary to note only that the lengths involved must be directed lengths; that is, if the direction from A to D is positive, then the length AD is positive but the length DB must be taken as negative.

The fact that any line intersecting the four lines OA, OB, OC and OD contains segments possessing the same cross ratio as the original segments suggests that we assign to the four projection lines meeting in the point O a particular cross ratio, namely the cross ratio of the segments on any section. Moreover, the

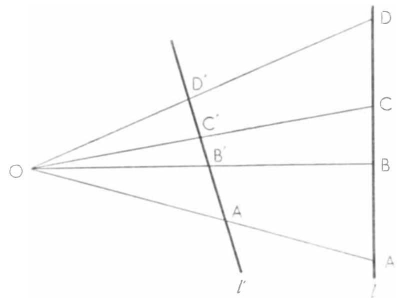


Figure 5

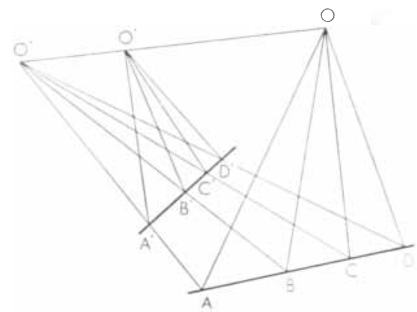


Figure 6

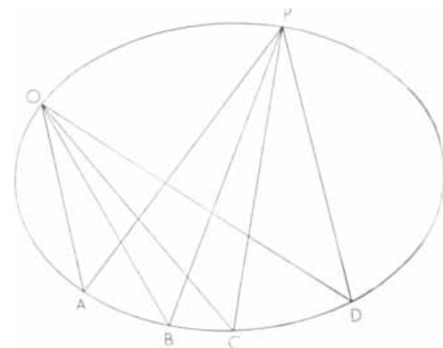


Figure 7

cross ratio of the four lines is a projective invariant, that is, if a projection of these four lines is formed and a section made of this projection, the section will contain four concurrent lines whose cross ratio is the same as that of the original four [*Figure 6*]. Here in the section *O′A′B′C′D′*, formed in the projection of the figure *OABCD* from the point *O″*, the four lines *O′A′*, *O′B′*, *O′C′* and *O′D′* have the same cross ratio as *OA*, *OB*, *OC* and *OD*.

The projective invariance of cross ratio was put to extensive use by the 19th-century geometers. We noted earlier in connection with Pascal's theorem that under projection and section a circle may become an ellipse, a hyperbola or a parabola, that is, any one of the conic sections. The geometers sought some common property which would account for the fact that a conic section always gave rise to a conic section, and they found the answer in terms of cross ratio. Given the points *O*, *A*, *B*, *C*, *D*, and a sixth point *P* on a conic section containing the others [*Figure 7*], then a remarkable theorem of projective geometry states that the lines *PA*, *PB*, *PC* and *PD* have the same cross ratio as *OA*, *OB*, *OC* and *OD*. Conversely, if *P* is any point such that *PA*, *PB*, *PC*, and *PD* have the same cross ratio as *OA*, *OB*, *OC* and *OD*, then *P* must lie on the conic through *O*, *A*, *B*, *C* and *D*. The essential point of this theorem and its converse is that a conic section is determined by the property of cross ratio. This new characterization of a conic was most welcome, not only because it utilized a projective property but also because it opened up a whole new line of investigation on the theory of conics.

The satisfying accomplishments of projective geometry were capped by the discovery of one of the most beautiful principles of all mathematics—the principle of duality. It is true in projective geometry, as in Euclidean geometry, that any two points determine one line, or as we prefer to put it, any two points lie on one line. But it is also true in projective geometry that any two lines determine, or lie on, one point. (The reader who has refused to accept the convention that parallel lines in Euclid's sense are also to be regarded as having a point in common will have to forego the next few paragraphs and pay for his stubbornness.) It will be noted that the second statement can be obtained from the first merely by interchanging the words point and line. We say in projective geometry that we have dualized the original statement. Thus we can speak not only of a set of points on a line but also of a set of lines on a point [*Figure 8*]. Likewise the dual of the figure consisting of four points no three of which lie on the same line is a figure of four lines no three of which lie on the same point [*Figure 9*].

Let us attempt this rephrasing for a slightly more complicated figure. A triangle consists of three points not all on the same line and the lines joining these points. The dual statement would read: three lines not all on the same point and the points joining them (that is, the points in which the lines intersect). The figure we get by rephrasing the definition of a triangle is again a triangle, and so the triangle is called self-dual.

Now let us rephrase Desargues's theorem in dual terms, using the fact that the dual of a triangle is a triangle and assuming in this case that the two triangles and the point *O* lie in one plane. The theorem says:

"If we have two triangles such that lines joining corresponding vertices pass through one point *O*, then the pairs of corresponding sides of the two triangles join in three points lying on one straight line."

Its dual reads:

"If we have two triangles such that points which are the joins of corresponding sides lie on one line *O*, then the pairs

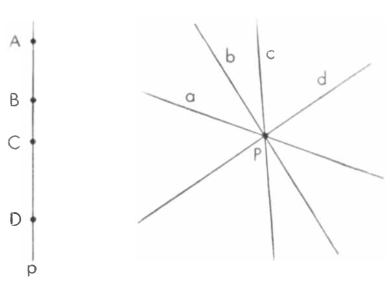


Figure 8

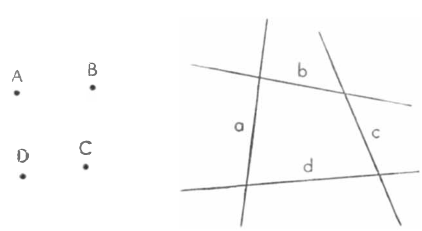


Figure 9

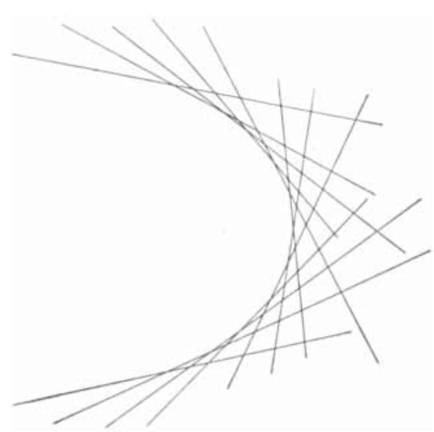


Figure 10

of corresponding vertices of the two triangles are joined by three lines lying on one point."

We see that the dual statement is really the converse of Desargues's theorem, that is, it is the result of interchanging his hypothesis with his conclusion. Hence by interchanging point and line we have discovered the statement of a new theorem. It would be too much to ask that the proof of the new theorem should be obtainable from the proof of the old one by interchanging point and line. But if it is too much to ask, the gods have been generous beyond our merits, for the new proof can be obtained in precisely this way.

Projective geometry also deals with curves. How should one dualize a statement involving curves? The clue lies in the fact that a curve is after all but a collection of points; we may think of a figure dual to a given curve as a collection of lines. And indeed a collection of lines which satisfies the condition dual to that satisfied by a conic section turns out to be the set of tangents to that curve [*Figure 10*]. If the conic section is a circle, the dual figure is the collection of tangents to the circle [*Figure 11*]. This collection of tangents suggests the circle as well as does the usual collection of points, and we shall call the collection of tangents the line circle.

Let us now dualize Pascal's theorem on the hexagon in a circle. His theorem goes:

"If we take six points, A, B, C, D, E and F, on the point circle, then the lines which join A and B and D and E join in a point P; the lines which join B and C and E and F join in a point Q; the lines which join C and D and F and A join in a point R. The three points P, Q and R lie on one line l."

Its dual reads:

"If we take six lines, a, b, c, d, e and f, on the line circle, then the points which join a and b and d and e are joined by the line p; the points which join b and c and e and f are joined by the line q; the points which join c and d and f and a are joined by the line r. The three lines p, q and r lie on one point L."

The geometric meaning of the dual statement amounts to this: Since the line circle is the collection of tangents to the point circle, the six lines on the line circle are any six tangents to the point circle, and these six tangents form a hexagon circumscribed about the point circle. Hence the dual statement tells us that if we circumscribe a hexagon about a point circle, the lines joining opposite vertices of the hexagon, lines p, q and r in the dual statement, meet in one point [*Figure 12*]. This dual statement is indeed a theorem of projective geometry. It is called Brianchon's theorem, after Monge's student Charles Brianchon, who discovered it by applying the principle of duality to Pascal's theorem pretty much as we have done.

It is possible to show by a single proof that every rephrasing of a theorem of projective geometry in accordance with the principle of duality must lead to a new theorem. This principle is a remarkable possession of projective geometry. It reveals the symmetry in the roles that point and line play in the structure of that geometry. The principle of duality also gives us insight into the process of creating mathematics. Whereas the discovery of this principle, as well as of theorems such as Desargues's and Pascal's, calls for imagination and genius, the discovery of new theorems by means of the principle is an almost mechanical procedure.

As one might suspect, projective geometry turns out to be more fundamental than Euclidean geometry. The clue to the relationship between the two geometries may be obtained by again considering projection and section. Consider the projection of a rectangle and a

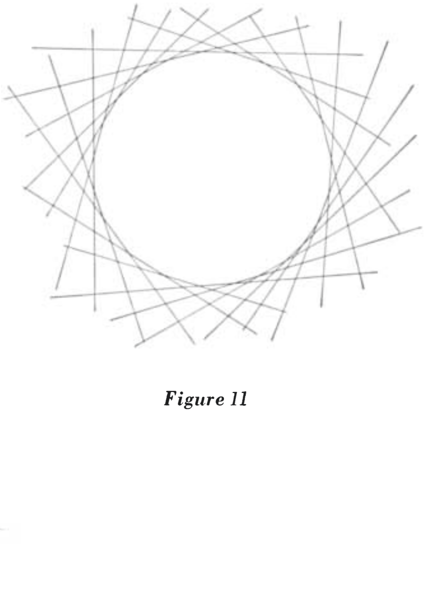


Figure 11

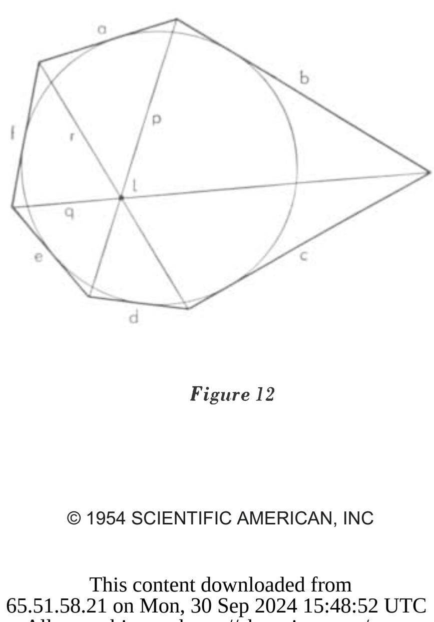


Figure 12

section in a plane parallel to the rectangle [*Figure 13*]. The section is a rectangle similar to the original one. If now the point *O* moves off indefinitely far to the left, the lines of the projection come closer and closer to parallelism with each other. When these lines become parallel and the center of the projection is the "point at infinity," the rectangles become not merely similar but congruent [*Figure 14*]. In other words, from the standpoint of projective geometry the relationships of congruence and similarity, which are so intensively studied in Euclidean geometry, can be studied through projection and section for special projections.

If projective geometry is indeed logically fundamental to Euclidean geometry, then all the concepts of the latter geometry should be defined in terms of projective concepts. However, in projective geometry as described so far there is a logical blemish: our definition of cross ratio, and hence concepts based on cross ratio, rely on the notion of length, which should play no role in projective geometry proper because length is not an invariant under arbitrary projection and section. The 19th-century geometer Felix Klein removed this blemish. He showed how to define length as well as the size of angles entirely in terms of projective concepts. Hence it became possible to affirm that projective geometry was indeed logically prior to Euclidean geometry and that the latter could be built up as a special case. Both Klein and Arthur Cayley even showed that the basic non-Euclidean geometries could be derived as special cases of projective geometry. No wonder that Cayley exclaimed: "Projective geometry is all geometry!"

It remained only to deduce the theorems of Euclidean and non-Euclidean geometry from axioms of projective geometry, and this geometers succeeded in doing in the late 19th and early 20th centuries. What Euclid did to organize the work of three hundred years preceding his time, the projective geometers did recently for the investigations which Desargues and Pascal initiated.

Research in projective geometry is now less active. Geometers are seeking to find simpler axioms and more elegant proofs. Some research is concerned with projective geometry in n-dimensional space. A vast new allied field is projective differential geometry, concerned with local or infinitesimal properties of curves and surfaces.

Projective geometry has had an important bearing on current mathematical research in several other fields. Projection and section amount to what is called in mathematics a transformation, and it seeks invariants under this transformation. Mathematicians asked: Are there other transformations more general than projection and section whose invariants might be studied? In recent times one new geometry has been developed by pursuing this line of thought, namely, topology. It would take us too far afield to consider topological transformations. It must suffice here to state that topology considers transformations more general than projection and section and that it is now clear that topology is logically prior to projective geometry. Cayley was too hasty in affirming that projective geometry is all geometry.

The work of the projective geometers has had an important influence on modern physical science. They prepared the way for the workers in the theory of relativity, who sought laws of the universe that were invariant under transformation from the coordinate system of one observer to that of another. It was the projective geometers and other mathematicians who invented the calculus of tensors, which proved to be the most convenient means for expressing invariant scientific laws.

It is of course true that the algebra of differential equations and some other branches of mathematics have contributed more to the advancement of science than has projective geometry. But no branch of mathematics competes with projective geometry in originality of ideas, coordination of intuition in discovery and rigor in proof, purity of thought, logical finish, elegance of proofs and comprehensiveness of concepts. The science born of art proved to be an art.

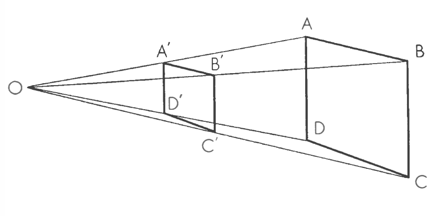


Figure 13

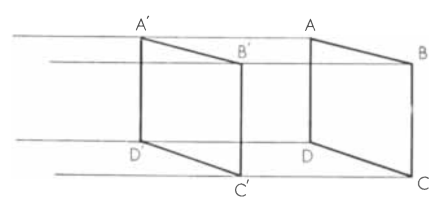


Figure 14