The Mathematical Experience
by Philip Davis & Reuben Hersh
Birkhäuser, 1981

sire is fulfilled? Or is it merely the wisdom of the Monday morning quarterback? Do these ideas work out in the classroom? Followups of attempts to reduce Pólya's program to practical pedagogics are difficult to interpret. There is more to teaching, apparently, than a good idea from a master.

Further Readings. See Bibliography

I. Goldstein and S. Papert; E. B. Hunt; A. Koestler [1964]; J. Kestin; G. Pólya [1945], [1954], [1962]; A. H. Schoenfeld; J. R. Slagle

The Creation of New Mathematics:

An Application of the Lakatos Heuristic

IN *PROOFS AND REFUTATIONS,* Imre Lakatos presents a picture of the "logic of mathematical discovery." A teacher and his class are studying the famous Euler-Descartes formula for polyhedra

$$V - E + F = 2.$$

Leonhard Euler
1707–1783

In this formula V is the number of vertices of a polyhedron, E, the number of its edges, and F, the number of its faces. Among the familiar polyhedra, these quantities take the following values:

	V	E	F
tetrahedron	4	6	4
(Egyptian) pyramid	5	8	5
cube	8	12	6
octahedron	6	12	8

(See also Chapter 7, Lakatos and the Philosophy of Dubitability.)

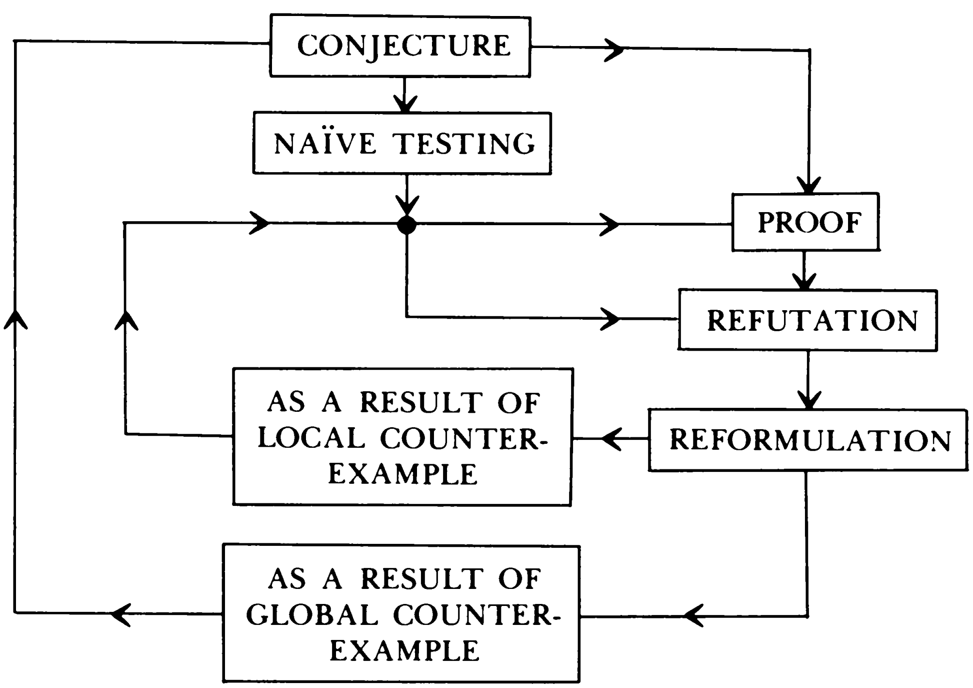
The teacher presents the traditional proof in which the polyhedron is stretched on the plane. This "proof" is immediately followed by a barrage of counterexamples presented by the students. Under the impact of these counterexamples, the statement of the theorem is modified, the proof is corrected and elaborated. New counterexamples are produced, new adjustments are made.

This development is presented by Lakatos as a model for the development of mathematical knowledge in general.

The Lakatos heuristic example of proofs and refutations which was formulated for the mathematical culture at large can of course be applied by the individual in his attempt to create new mathematics. The writer has used the method with moderate success in his classes. The initial shock of presenting students not with a fixed problem to be cracked, but with an open-ended situation of potential discovery, must and can be overcome. The better students then experience a sense of exhilaration and freedom in which they are in control of the material.

I shall illustrate the method with a little example from the elementary theory of numbers.

Simplified Lakatos model for the heuristics of mathematical discovery



I begin with an initial statement which I shall call the "seed." The seed statement should be an interesting one, quite simple. The object of the exercise is for the student to water the seed so that it grows into a sturdy plant. I usually present the class with a variety of seeds and they select them for watering, depending upon their experience.

Act I

Seed: "If a number ends in 2, it is divisible by 2."

Examples: 42 ends in and is divisible by 2. 172 ends in and is divisible by 2.

Proof: A number is even if and only if it ends in 0, 2, 4, 6, 8. All even numbers are divisible by 2. In particular, those that end in 2 are divisible by 2.

Proof (more sophisticated): If the number, in digit form, is $ab \ldots c2$, then it is clearly of the form $(ab \ldots c0) + 2$, hence of the form $10Q + 2 = 2(5Q + 1)$.

Conjectural Leap: If a number ends in N it is divisible by N.

Comment: Be bold and make the obvious generalization. The heavens won't fall in if it turns out to be false.

Example: If a number ends in 5 it is divisible by 5. Sure: 15, 25, 128095, etc. But, alas,

Counterexample: If a number ends in 4 is it divisible by 4? Is 14 divisible by 4? No. Too bad.

Objection: But some numbers that end in 4 *are* divisible by 4: 24. Some numbers ending in 9 *are* divisible by 9: 99.

Recap of Experience: The numbers 1, 2, . . . , 9 seem to be divisible into two categories. *Category I:* The digits N such that when a number ends in N it is divisible by N, always. *Category II:* The digits N such that when a number ends in N it is divisible by N only occasionally.
Category I: 1, 2, 5.
Category II: 3, 4, 6, 7, 8, 9.

Point of Order: What about numbers that end in 0? Are they divisible by 0? No. But they are divisible by 10. Hmm.

We may have to watch this. This phenomenon doesn't fit in with the form of the seed.

Definition: Let's call the numbers in Category I "magic numbers." They have a delightful property.

Tentative Theorem: The numbers 1, 2 and 5 are magic numbers. They are the only magic numbers.

Counterexample: What about the number 25? Isn't it magic? If a number ends in 25 it is divisible by 25. For example: 225, or 625.

Objection: We thought you were talking about single-digit numbers.

Rebuttal: Well, we were originally. But the 25 phenomenon is interesting. Let's open up the original inquiry a bit.

Reformulation: Let N now represent not necessarily a single digit but a whole group of digits like 23, 41, 505, etc. Make the definition that N is magic if a number that ends with the digit group N is divisible by N. Does this extended definition make sense?

Examples: Yes it does. 25 is magic. 10 is magic. 20 is magic. 30 is magic.

Counterexample: 30 isn't magic. 130 is not divisible by 30. Come to think of it, how do you know 25 is magic?

Theorem: 25 is a magic number.

Proof: If a number ends in 25 it is of the digital form $abc \ldots e25 = abc \ldots e00 + 25$, hence of the form $100Q + 25 = 25(4Q + 1)$.

Reformulation of Goal: Find all the magic numbers.

Accumulation of Experience: 1, 2, 5, 10, 25, 50, 100, 250, 500, 1000 are all magic numbers.

Observation: All the magic numbers we have been able to find seem to be products of 2's and 5's. Certainly the ones in the above list are.

Conjecture: Any number N of the form $N = 2^p \cdot 5^q$ where $p \geq 0$, $q \geq 0$ is a magic number.

Comment: Seems reasonable. What have we got to lose?

Counterexample: Take $p = 3$, $q = 1$. Then $N = 2^3 \cdot 5 = 40$. Is a number that ends in 40 always divisible by 40? No. E.g., 140.

Reformulation: How about the other way around, though? All the magic numbers we have found are of the form $2^p \cdot 5^q$. Perhaps all magic numbers are of that form.

Objection: Isn't that what you just proposed?

Rebuttal: No, what was proposed was the other way around: a number of the form $2^p \cdot 5^q$ is magic. See the difference?

Theorem: If N is a magic number then $N = 2^p \cdot 5^q$.

Proof: Let a number end in N (recall: in this statement N is acting as a group of digits.) Then the number looks like $abc \ldots eN$, digitwise. We would like to split it up, as before. Therefore, let N have $d(N)$ digits. Then the number $abc \ldots eN$ is really $abc \ldots e00 \ldots 0 + N$ where there are $d(N)$ 0's at the end. Therefore the number is of the form $Q \cdot 10^{d(N)} + N$. (Try this out when $d(N) = 2, 3$, etc.) All numbers that end with N are of this form. Conversely, if Q is any number whatever then the number $Q \cdot 10^{d(N)} + N$ ends with N. Now if N is magic it always divides $Q \cdot 10^{d(N)} + N$. Since N divides N, it must always divide $Q \cdot 10^{d(N)}$ for all Q. But Q might be the simple number 1, for example. Therefore N must divide $10^{d(N)}$. Since $10^{d(N)} = 2^{d(N)} \cdot 5^{d(N)}$ is a prime factorization, it follows that N must itself factor down to a certain number of 2's and 5's.

Current Position: We now know that a magic number is one of the form $N = 2^p \cdot 5^q$ for some integers $p, q \geq 0$. We would like to turn it around. Then we should have a necessary and sufficient condition for magicality.

Refocussing of Experience: Since we know that all magic numbers are of the form $N = 2^p \cdot 5^q$, the problem comes down to: what must be asserted about p and q to make the resulting N magic?

Conjecture: $p \leq q$?

Counterexample: $p = 0$, $q = 4$, $N = 2^0 \cdot 5^4 = 625$. Is 625 magic? No: 1625 doesn't divide by 625.

Conjecture: $p = q$?

Objection: Then $N = 2^p 5^p = 10^p$ or 1, 10, 100, ... O.K. But there are other magic numbers.

Conjecture: $p \geq q$?

Counterexample: $p = 3$, $q = 1$, $N = 2^3 \cdot 5^1 = 40$. This is not magic.

Observation: Hmm. Something subtle at work here. This brings down the curtain on Act I. The process goes on for those with sufficient interest and strength.

Act II

(In this act, the heuristic line is severely abbreviated in the write-up.)

Strategy Conference: Let's go back to the proof of the necessity of the form $N = 2^p \cdot 5^q$. We found that if N is magic it divides $10^{d(N)}$. Recall that $d(N)$ stands for the number of digits in the group of digits N. Perhaps this is sufficient as well? Aha! A breakthrough?

Theorem: N is magic if and only if it divides $10^{d(N)}$.

Proof: The necessity has already been proved. If a number ends in N, then, as we know, it is of the form $Q \cdot 10^{d(N)} + N$. But N divides N and N is assumed to divide $10^{d(N)}$. Therefore it surely divides $Q \cdot 10^{d(N)} + N$.

Aesthetic Objection: While it is true that we now have a necessary and sufficient condition for magicality, this condition is on N itself and not on its factored form $2^p \cdot 5^q$.

Conference: When does $N = 2^p \cdot 5^q$ divide $10^{d(N)}$? Well, $10^{d(N)} = 2^{d(N)} \cdot 5^{d(N)}$, so that obviously a necessary and sufficient condition for this is $p \leq d(N)$, $q \leq d(N)$. But this is equivalent to $\max(p, q) \le d(N)$. We still have the blasted $d(N)$ to contend with. We don't want it. We'd like a condition on N itself, or possibly on p and q. How can we convert $\max(p, q) \le d(N) = d(2^p \cdot 5^q)$ into a more convenient form? As we know, $p = q$ is O.K. Let's see this written in the new form: $p = \max(p, p) \le d(2^p \cdot 5^p) = d(10^p)$. Now the number of digits in 10^p is $p + 1$. So this is saying $p \le p + 1$ which is O.K. What if, in the general case, we "even out" the powers of 2 and the powers of 5? Write $q = p + h$ where $h > 0$. (Aha!)

Objection: What if $p > q$ so that $q = p + h$ is impossible with $h > 0$?

Rebuttal: Treat that later.

Conference: $\max(p, p + h) \le d(2^p \cdot 5^{p+h}) = d(2^p \cdot 5^p \cdot 5^h) = d(10^p \cdot 5^h)$. Now since $h > 0$, $\max(p, p + h) = p + h$. Also, the number of digits in $10^p \cdot Q$ where Q is any number $= p +$ number of digits in Q. Therefore $p + h \le p + d(5^h)$ or: $h \le d(5^h)$.

Query: When is it true that $h > 0$ and $h \le d(5^h)$?

Experimentation: $h = 1$: $1 \le d(5^1)$ O.K. $h = 2$: $2 \le d(5^2)$ O.K. $h = 3$: $3 \le d(5^3)$ O.K. $h = 4$: $4 \le d(5^4) = d(625) = 3$. No good. $h = 5$: $5 \le d(5^5) = d(3125) = 4$. No good.

Conjecture: $h \le d(5^h)$ if and only if $h = 1, 2, 3$.

Proof: Omitted.

Reprise: What about $p > q$?

Conference: Set $p = q + h$, $h > 0$. $q + h = \max(q + h, q) \le d(2^{q+h} \cdot 5^q) = d(10^q \cdot 2^h) = q + d(2^h)$, or $h \le d(2^h)$. When is $h \le d(2^h)$?

Experimentation: $h = 1 : 1 \le d(2^1)$ O.K. $h = 2 : 2 \le d(2^2)$ No good.

Conjecture: $h \le d(2^h)$ if and only if $h = 1$.

Proof: Omitted.

Theorem: *N* is magic if and only if it equals a power of ten times 1, 2, 5, 25, or 125.

Proof: Omitted.

In anticipation of further developments we might like to write this theorem in a different way.

Theorem: *N* is magic if and only if $N = 2^p \cdot 5^q$, where $0 \leq q - p + 1 \leq 4$.

Proof: Omitted.

Act III might begin by asking what would happen if we wrote our numbers in some base other than 10. What about a prime base, or a base equal to a power of a prime?

Further Readings. See Bibliography

M. Gardner, U. Grenander; I. Lakatos, [1976].

Comparative Aesthetics

WHAT ARE THE ELEMENTS that make for creativity? Is it a deep analytic ability deriving from ease of combinatorial or geometric visualization—a mind as restless as a swarm of honeybees in a garden, flitting from fact to fact, perception to perception and making connections, aided by a prodigious memory—a mystic intuition of how the universe speaks mathematics—a mind that operates logically like a computer, creating implications by the thousands until an appropriate configuration emerges?

Or is it some extralogical principle at work, a grasp and a use of metaphysical principles as a guide? Or, as Henri Poincaré thought, a deep appreciation of mathematical aesthetics?