The Mathematical Experience
by Philip Davis & Reuben Hersh
Birkhäuser, 1981

Pólya's Craft of Discovery

"My mind was struck by a flash of lightning in which its desire was fulfilled."

Dante, *Paradiso,* Canto XXXIII
Quoted by G. Pólya

GEORGE PÓLYA (1888–) has had a scientific career extending more than seven decades. A brilliant mathematician who has made fundamental contributions in many fields, Pólya has also been a brilliant teacher, a teacher's teacher, and an expositor. Pólya believes that there is a craft of discovery. He believes that the ability to discover and the ability to invent can be enhanced by skillful teaching which alerts the student to the principles of discovery and which gives him an opportunity to practise these principles.

George Pólya
1888–1985

In a series of remarkable books of great richness, the first of which was published in 1945, Pólya has crystallized these principles of discovery and invention out of his vast experience, and has shared them with us both in precept and in example. These books are a treasure-trove of strategy, know-how, rules of thumb, good advice, anecdote, mathematical history, together with problem after problem at all levels and all of unusual mathematical interest. Pólya places a global plan for "How to Solve It" in the endpapers of his book of that name:

HOW TO SOLVE IT

First: you have to *understand* the problem.

Second: find the connection between the data and the unknown. You may be obliged to consider auxiliary problems if an immediate connection cannot be found. You should obtain eventually a *plan* of the solution.

Third: *Carry out* your plan.

Fourth: *Examine* the solution obtained.

These precepts are then broken down to "molecular" level on the opposite endpaper. There, individual strategies are suggested which might be called into play at appropriate moments, such as

> If you cannot solve the proposed problem, look around for an appropriate related problem
> Work backwards
> Work forwards
> Narrow the condition
> Widen the condition
> Seek a counterexample
> Guess and test
> Divide and conquer
> Change the conceptual mode

Each of these heuristic principles is amplified by numerous appropriate examples.

Subsequent investigators have carried Pólya's ideas forward in a number of ways. A. H. Schoenfeld has made an interesting tabulation of the most frequently used heuristic principles in college-level mathematics. We have appended it here.

*FREQUENTLY USED HEURISTICS**

Analysis

1) DRAW A DIAGRAM if at all possible.
2) EXAMINE SPECIAL CASES:
   a) Choose special values to exemplify the problem and get a "feel" for it.
   b) Examine limiting cases to explore the range of possibilities.
   c) Set any integer parameters equal to 1, 2, 3, . . . , in sequence, and look for an inductive pattern.
3) TRY TO SIMPLIFY THE PROBLEM by
   a) exploiting symmetry, or
   b) "Without Loss of Generality" arguments (including scaling)

**From: A. H. Schoenfeld*

Exploration

1) CONSIDER ESSENTIALLY EQUIVALENT PROBLEMS:
   a) Replacing conditions by equivalent ones.
   b) Re-combining the elements of the problem in different ways.
   c) Introduce auxiliary elements.
   d) Re-formulate the problem by
      i) change of perspective or notation
      ii) considering argument by contradiction or contrapositive
      iii) assuming you have a solution, and determining its properties
2) CONSIDER SLIGHTLY MODIFIED PROBLEMS:
   a) Choose subgoals (obtain partial fulfillment of the conditions)
   b) Relax a condition and then try to re-impose it.
   c) Decompose the domain of the problem and work on it case by case.
3) CONSIDER BROADLY MODIFIED PROBLEMS:
   a) Construct an analogous problem with fewer variables.
   b) Hold all but one variable fixed to determine that variable's impact.
   c) Try to exploit any related problems which have similar
      i) form
      ii) "givens"
      iii) conclusions.

Remember: when dealing with easier related problems, you should try to exploit both the RESULT and the METHOD OF SOLUTION on the given problem.

Verifying your solution

1) DOES YOUR SOLUTION PASS THESE SPECIFIC TESTS:
   a) Does it use all the pertinent data?
   b) Does it conform to reasonable estimates or predictions?
   c) Does it withstand tests of symmetry, dimension analysis, or scaling?
2) DOES IT PASS THESE GENERAL TESTS?
   a) Can it be obtained differently?
   b) Can it be substantiated by special cases?
   c) Can it be reduced to known results?
   d) Can it be used to generate something you know?

To give the flavor of Pólya's thinking and writing in a very beautiful but subtle case, a case that involves a change in the conceptual mode, I shall quote at length from his *Mathematical Discovery* (vol. II, pp. 54 ff):

Example

I take the liberty of trying a little experiment with the reader. I shall state a simple but not too commonplace theorem of geometry, and then I shall try to reconstruct the sequence of ideas that led to its proof. I shall proceed slowly, very slowly, revealing one clue after the other, and revealing each clue gradually. I think that before I have finished the whole story, the reader will seize the main idea (unless there is some special hampering circumstance). But this main idea is rather unexpected, and so the reader may experience the pleasure of a little discovery.

A. *If three circles having the same radius pass through a point, the circle through their other three points of intersection also has the same radius.*

This is the theorem that we have to prove. The statement is short and clear, but does not show the details distinctly enough. If we *draw a figure* (Fig. 10.1) and *introduce suitable notation,* we arrive at the following more explicit restatement:

B: *Three circles k, l, m have the same radius r and pass through the same point O. Moreover, l and m intersect in the point A, m and k in B, k and l in C. Then the circle e through A, B, C has also the radius r.*

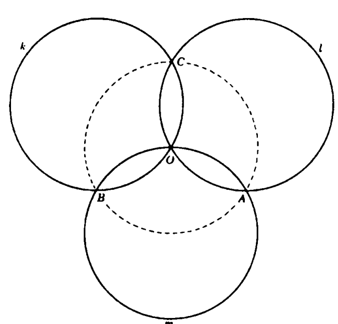


Fig. 10.1. Three circles through one point.

Figure 10.1 exhibits the four circles *k*, *l*, *m*, and *e* and their four points of intersection *A*, *B*, *C*, and *O*. The figure is apt to be unsatisfactory, however, for it is not simple, and it is still incomplete; something seems to be missing; we failed to take into account something essential, it seems.

We are dealing with circles. What is a circle? A circle is determined by center and radius; all its points have the same distance, measured by the length of the radius, from the center. We failed to introduce the common radius *r*, and so we failed to *take into account an essential part of the hypothesis.* Let us, therefore, introduce the centers, *K* of *k*, *L* of *l*, and *M* of *m*. Where should we exhibit the radius *r*? There seems to be no reason to treat any one of the three given circles k, l, and m or any one of the three points of intersection A, B, and C better than the others. We are prompted to connect all three centers with all the points of intersection of the respective circle: K with B, C, and O, and so forth.

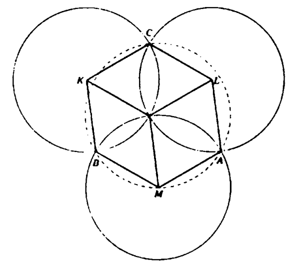


Fig. 10.2. Too crowded.

The resulting figure (Fig. 10.2) is disconcertingly crowded. There are so many lines, straight and circular, that we have much trouble in "seeing" the figure satisfactorily; it "will not stand still." It resembles certain drawings in old-fashioned magazines. The drawing is ambiguous on purpose; it presents a certain figure if you look at it in the usual way, but if you turn it to a certain position and look at it in a certain peculiar way, suddenly another figure flashes on you, suggesting some more or less witty comment on the first. Can you recognize in our puzzling figure, overladen with straight lines and circles, a second figure that makes sense?

....

We may hit in a flash on the right figure hidden in our overladen drawing, or we may recognize it gradually. We may be led to it by the effort to solve the proposed problem, or by some secondary, unessential circumstance. For instance, when we are about to redraw our unsatisfactory figure, we may observe that the *whole* figure is determined by its *rectilinear part* (Fig. 10.3).

This observation seems to be significant. It certainly simplifies the geometric picture, and it possibly improves the logical situation. It leads us to restate our theorem in the following form.

C. *If the nine segments*

$$\begin{array}{ccc} KO, & KC, & KB, \\ LC, & LO, & LA, \\ MB, & MA, & MO, \end{array}$$

are all equal to r, there exists a point E such that the three segments

$$EA, \qquad EB, \qquad EC$$

are also equal to r.

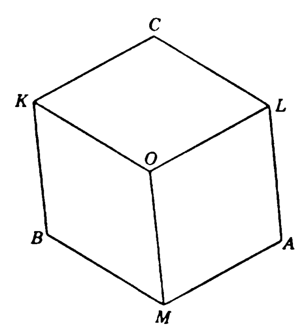


Fig. 10.3. It reminds you—of what?

This statement directs our attention to Fig. 10.3. This figure is attractive; it reminds us of something familiar. (Of what?)

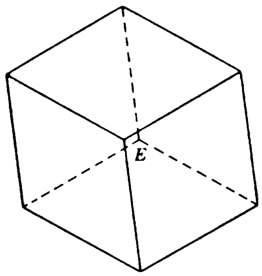


Fig. 10.4. Of course!

Of course, certain quadrilaterals in Fig. 10.3, such as *OLAM* have, by hypothesis, four equal sides, they are rhombi. A rhombus is a familiar object; having recognized it, we can "see" the figure better. (Of what does the *whole* figure remind us?)

Opposite sides of a rhombus are parallel. Insisting on this remark, we realize that the 9 segments of Fig. 10.3 are of three kinds; segments of the same kind, such as *AL*, *MO*, and *BK*, are parallel to each other. (Of what does the figure remind us *now?*)

We should not forget the conclusion that we are required to attain. Let us assume that the conclusion is true. Introducing into the figure the center *E* or the circle *e*, and its three radii ending in *A*, *B*, and *C*, we obtain (supposedly) still more rhombi, still more parallel segments; see Fig. 10.4. (Of what does the whole figure remind us *now?*)

....

Of course, Fig. 10.4 is the projection of the 12 edges of a parallelepiped having the particularity that the projection of all edges are of equal length.

Figure 10.3 is the projection of a "nontransparent" parallelepiped; we see only 3 faces, 7 vertices, and 9 edges; 3 faces, 1 vertex, and 3 edges are invisible in this figure. Figure 10.3 is just a part of Fig. 10.4, but this part defines the whole figure. If the parallelepiped and the direction of projection are so chosen that the projections of the 9 edges represented in Fig. 10.3 are all equal to *r* (as they should be, by hypothesis), the projections of the 3 remaining edges must be equal to *r*. These 3 lines of length *r* are issued from the projection of the 8th, the invisible vertex, and this projection *E* is the center of a circle passing through the points *A*, *B*, and *C*, the radius of which is *r*.

Our theorem is proved, and proved by a surprising, artistic conception of a plane figure as the projection of a solid.

(The proof uses notions of solid geometry. I hope that this is not a great wrong, but if so it is easily redressed. Now that we can characterize the situation of the center *E* so simply, it is easy to examine the lengths *EA*, *EB*, and *EC* independently of any solid geometry. Yet we shall not insist on this point here.)

This is very beautiful, but one wonders. Is this the "light that breaks forth like the morning," the flash in which desire is fulfilled? Or is it merely the wisdom of the Monday morning quarterback? Do these ideas work out in the classroom? Followups of attempts to reduce Pólya's program to practical pedagogics are difficult to interpret. There is more to teaching, apparently, than a good idea from a master.

Further Readings. See Bibliography

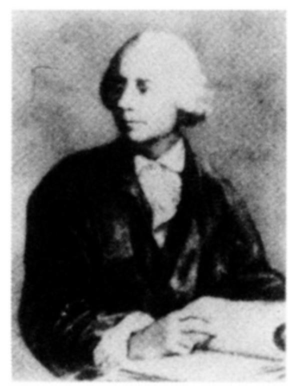
I. Goldstein and S. Papert; E. B. Hunt; A. Koestler [1964]; J. Kestin; G. Pólya [1945], [1954], [1962]; A. H. Schoenfeld; J. R. Slagle

The Creation of New Mathematics:

An Application of the Lakatos Heuristic

IN *PROOFS AND REFUTATIONS,* Imre Lakatos presents a picture of the "logic of mathematical discovery." A teacher and his class are studying the famous Euler-Descartes formula for polyhedra

$$V - E + F = 2.$$



Leonhard Euler
1707–1783

In this formula V is the number of vertices of a polyhedron, E, the number of its edges, and F, the number of its faces. Among the familiar polyhedra, these quantities take the following values:

	V	E	F
tetrahedron	4	6	4
(Egyptian) pyramid	5	8	5
cube	8	12	6
octahedron	6	12	8

(See also Chapter 7, Lakatos and the Philosophy of Dubitability.)